L^p BOUNDS FOR PSEUDO-DIFFERENTIAL OPERATORS

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ABSTRACT

Sharp L^p boundedness results are proven for pseudo-differential operators in the class $S^m_{\rho\delta}$.

We show how some recent results in [1] yield sharp L^p estimates for pseudo-differential operators in Hörmander's class $S^m_{\rho\delta}$. Recall that pseudo-differential operators have the form $Tf(x) = \sigma(x, D)f(x) = \int_{R^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)d\xi$, where $\hat{}$ denotes the Fourier transform.

One imposes restrictions on the symbol σ to get a manageable calculus of operators wide enough to include inverses of interesting differential operators. For example, to invert the heat operator in a class of pseudo-differential operators invariant under C^∞ changes of coordinates, Hörmander posed the following condition.

Definition. A symbol $\sigma(x,\xi)$ belongs to $S^m_{\rho\delta}$ if

$$\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\alpha}{\partial \xi^\alpha}\sigma(x,\xi) = O(|\xi|^{m-|\alpha|\rho+|\beta|\delta}),$$

as $|\xi| \to \infty$, for all multi-indices α, β.

The *norm* of a symbol in $S^m_{\rho\delta}$ is

$$\|\sigma\|_S = \sup_{\substack{|\alpha|\leq k\\|\beta|\leq N}} \left|\frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\alpha}{\partial \xi^\alpha}\sigma(x,\xi)\right|(1+|\xi|)^{-m+|\alpha|\rho-|\beta|\delta},$$

where $k, N > n/2$.

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An operator $T = \sigma(x,D)$ with $\sigma \in S^m_{\rho\delta}$ has order m on L^2 for $0 \leqq \delta < \rho < 1$. (cf. [3]). That is, T: $L^2_s(R^n) \to L^2_{s-m}(R^n)$, where L^p_s is the space of functions (or distributions) with s'th derivatives in L^p. Since $|\sigma(x,\xi)| = O(|\xi|^m)$, this is entirely reasonable. However, a remarkable example of Hardy-Littlewood [7], Hirschman [2] and Wainger [6] shows that on $L^p(p \neq 2)$, $\sigma(x,D)$ can have "order" $m' = m'(p,\delta,p,m)$ strictly greater than m. In other words T maps L^p_s not to L^p_{s-m}, but only to $L^p_{s-m'}$. Our purpose here is to prove a sharp result on the L^p orders of pseudo-differential operators.

THEOREM.

a) *Let $\sigma(x,\xi) \in S^{-\beta}_{1-a,\delta}(R^n)$ with $0 \leqq \delta < 1 - a < 1$ and $\beta < na/2$. Then $\sigma(x,D)$ is bounded on L^p for*

$$\left|\frac{1}{p} - \frac{1}{2}\right| \leqq \gamma = \frac{\beta}{n}\left[\frac{n/2+\lambda}{\beta+\lambda}\right], \ \lambda = \frac{na/2-\beta}{1-a}.$$

b) *If $|1/p - 1/2| > \gamma$, then the symbol*

$$\sigma(x,\xi) = \sigma_{\alpha\beta}(\xi) = \frac{e^{i|\xi|^\alpha}}{1+|\xi|^\beta} \in S^{-\beta}_{1-a,0}$$

provides an operator $\sigma_{\alpha\beta}(D)$ unbounded on L^p.

c) *Let $\sigma(x,\xi) \in S^{-na/2}_{1-a\delta}$, so that the critical L^p* space *is L^1. Although $\sigma(x,D)$ is unbounded on L^1, it is bounded on the Hardy space H^1. (See [5] for a definition of H^1 in n variables.)*

This result was partly known before. Part (b) is the counterexample of Hardy-Littlewood-Hirschman-Wainger. Part (a) was proved in the noncritical case $|1/p - 1/2| < \gamma$ by Hirschman and Wainger (for constant-coefficient symbols) and by Hörmander (for general symbols). However, for our purposes the basic real-variable property of $\sigma \in S^m_{\rho\delta}$ is part (c). From (c), (a) can be deduced in full strength from a (nontrivial) interpolation. The proofs of (a) and (c) require a new technique discovered recently by E. M. Stein and the author. At the heart of our method lies the interesting class of functions of bounded mean oscillation (B.M.O.) defined by F. John and L. Nirenberg in [4]. A real-valued function f on R^n belongs to BMO if the norm $\|f\|_{\text{BMO}} = \sup_Q 1/|Q| \int_Q |f(x) - av_Q f| dx$ is finite. Here Q denotes an arbitrary cube in R^n, and $av_Q f = 1/|Q| \int_Q f(x)dx$. For many purposes, BMO is a natural substitute for L^∞. The results in [1] reduce (a) and (c) above to the following

PROPOSITION. *Let* $\sigma(x,\xi)\in S^{-na/2}_{1-a,\delta}$ *for* $0 \leqq \delta < 1-a < 1$. *Then* $\sigma(x,D)$ *is a bounded operator from* L^∞ *to BMO.*

In what follows, we content ourselves with proving the proposition and refer the reader to [1] for the new technique of using BMO. For the constant-coefficient case $\sigma(x,\xi) = \sigma(\xi)$, [1] proves both the proposition and the theorem.

A simple special case of the Hardy-Littlewood-Sobolev theorem is useful in proving the proposition.

LEMMA. *For* $q(x,\xi)\in S^{-na/2}_{1-a,\delta}$ *supported entirely in* $|\xi| \leqq 1$ *or in* $r \leqq |\xi| \leqq 3r$, *we have* $\| q(x,D)f\|_\infty \leqq C\|q\|_S\|f\|_\infty$.

PROOF. Say q is supported in $r \leqq |\xi| \leqq 3r$. We have $q(x,D)f(x) = \hat{q}_x * f(x)$, where $\hat{q}_x$ is the Fourier transform of $q_x(\xi) = q(x,\xi)$. Therefore $|q(x,D)f(x)| \leqq \|\hat{q}_x\|_1\|f\|_\infty$, so that if we show that $\|\hat{q}_x\|_1 \leqq C\|q\|_S$, the proof will be complete. Let $b = r^{a-1}$. Then

$$\int_{|y|<b} |\hat{q}_x(y)|\,dy \leqq cb^{n/2}\left(\int_{|y|<b} \hat{q}_x(y)|^2 dy\right)^{1/2} \leqq cb^{n/2}\left(\int_{R^n}|q(x,\xi)|^2\,d\xi\right)^{1/2}$$

$$\leqq C\|q\|_S \text{ (since } q \text{ lives in } |\xi| \sim r),$$

and

$$\int_{|y|\leqq b} |\hat{q}_x(y)|\,dy \leqq cb^{\,n/2-k}\left(\int_{|y|\geqq b} |y|^{2k}|\hat{q}_x(y)|^2 dy\right)^{1/2}$$

$$\leqq cb^{\,n/2-k}\left(\int_{R_n} |\nabla^{(k)}_\xi q(x,\xi)|^2 d\xi\right)^{1/2} \leqq C\|q\|_S$$

(again since q lives in $|\xi| \sim r$). Thus $\|\hat{q}_x\|_1 \leqq C\|q\|_S$. Q.E.D.

Now we can prove our proposition. Fix $f\in L^\infty$ and $Q \subseteq R^n$ having side d and center x_0. We have to show

$$\text{(1)} \qquad \frac{1}{|Q|}\int_Q |\sigma(x,D)f(x) - a_Q|\,dx \leqq C\|f\|_\infty.$$

Break up $\sigma(x,\xi)$ into two parts, $\sigma = \sigma^0 + \sigma^1$, with σ^0 supported in $|\xi| \leqq 2d^{-1}$, σ^1 supported in $|\xi| \geqq d^{-1}$, and $\|\sigma^0\|_S$, $\|\sigma^1\|_S \leqq C\|\sigma\|_S$. Let us look first at σ^0.

We have $(\partial/\partial x_j)\sigma^0(x,D)f(x) = \sigma'(x,D)f(x)$, where σ' is the symbol $\sigma'(x,\xi) =(\partial/\partial x_j)\sigma^0(x,\xi) + \xi_j\sigma^0(x,\xi)$. Writing σ' in any reasonable way as $\sigma'(x,\xi) = \sum^j_{j\leqq 0}\rho_j(x,\xi)$ with ρ_j supported in $|\xi| \sim 2^{-j}d^{-1}$, we discover from elementary computation that $\|\rho_j\|_S \leqq Cd^{-1}2^{-jc}\|\sigma\|_S$. By the lemma,

$$\left\| \frac{\partial}{\partial x_j} \sigma^0(x,D)f \right\|_\infty \leqq \sum_{j \geqq 0} \| \rho_j(x,D)f \|_\infty \leqq Cd^{-1} \sum_{j \geqq 0} 2^{-jc} \| \sigma \|_S \| f \|_\infty$$

$$\leqq Cd^{-1} \| \sigma \|_S \| f \|_\infty .$$

Therefore, $|\sigma^0(x,D)f(x) - a_Q|$ remains bounded in Q for some constant a_Q, so that

(2) $$\frac{1}{|Q|} \int_Q |\sigma^0(x,D)f(x) - a_Q| \, dx \leqq C \| \sigma \|_S \| f \|_\infty .$$

This estimates the σ^0 contribution to (1).

To handle the σ^1 term, fix a "bump" function ϕ on R^n, with $0 \leqq \phi \leqq 10$, $\phi \geqq 1$ on Q and ϕ having "thickness" d^{1-a}. (Say $\hat{\phi}(\xi)$ is supported in $|\xi| \leqq d^{a-1}$.) Then

(3) $$\phi(x) \cdot \sigma^1(x,D)f(x) = \sigma^1(x,D)(\phi f)(x) + [\phi, \sigma^1(x,D)]f(x) \sim I + II.$$

To estimate I, write $\sigma^1(x,D)(\phi f) = (\sigma^1(x,D) \cdot J^{-na/2}) \cdot (J^{na/2}(\phi f))$ where J is a Bessel potential. $\sigma^1(x,D) \cdot J^{-na/2}$ is a pseudo-differential operator with symbol $\sigma^1(x,\xi) \cdot (1 + |\xi|)^{na/2} \in S^0_{1-a,\delta}$, so that by Hörmander's L^2 result,

$$\| \sigma^1(x,D)(\phi f) \|_2^2 \leqq C \| {}^1\sigma \|_S^2 \| J^{na/2}(\phi f) \|_2^2 \leqq C \| {}^1\sigma \|_S^2 \| f \|_\infty^2 |Q| ,$$

since obviously $J^{na/2}(\phi f) \leqq \| f \|_\infty \cdot J^{na/2}(|\phi|)$. Consequently,

(4) $$\frac{1}{|Q|} \int_Q |\sigma^1(x,D)(\phi f)(x)| \, dx \leqq \left(\frac{1}{|Q|} \int_Q |\sigma^1(x,D)(\phi f)(x)|^2 dx \right)^{1/2}$$

$$\leqq C \| \sigma \|_S \| f \|_\infty , \text{ which takes care of } I.$$

II can be written in the form $\theta(x,D)f(x)$, where θ is the symbol $\theta(x,\xi) = \int_{R^n} e^{ix \cdot \eta} \hat{\phi}(\eta) [\sigma^1(x,\xi) - \sigma^1(x, \xi - \eta)] d\eta$. Write $\theta = \sum_{j=0}^\infty \theta_j$ in any reasonable way, with $\theta_j(x,\xi)$ supported in $|\xi| \sim 2^j d^{-1}$. Elementary calculations show that $\| \theta_j \|_S \leqq C \cdot 2^{-je} \| \sigma \|_S$, so that by the lemma, $\| [\phi \sigma^1(x,D)] f \|_\infty \leqq \sum_{j=0}^\infty \| \theta_j(x,D) f \| \leqq \sum_{j=0}^\infty C \cdot 2^{-jc} \| \sigma \|_S \| f \|_\infty \leqq C \| \sigma \|_S \| f \|_\infty$. Putting this and (4) into (3) we obtain $1/|Q|) \int_Q |\phi(x) \cdot \sigma^1(x,D)f(x)| \, dx \leqq C \| \sigma \|_S \| f \|_\infty$. Since $|\phi(x)| \geqq 1$ on Q, we have $(1/|Q|) \int_Q |\sigma^1(x,D)f(x)| \, dx \leqq C \| \sigma \|_S \| f \|_\infty$. Together with (2), this proves (1). Q.E.D.

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